A Derivation of Fluid Mechanics

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I derive a single PDE that describes the dynamics of fluids in state space. Then the similarity between that PDE and equations of fluid mechanics is demonstrated by using it to deriving a set of three equations analogous to the mass, internal energy and Navier-Stokes equations. Finally I demonstrate that for a fluid with particles following the Maxwell-Boltzmann distribution the set of analogous equations reduces to the equations of inviscid flow.

I. INTRODUCTION

Fluids consist of a large number of interacting particles, presumably the fluid mechanics observed at a macroscopic level is a result of the interactions occurring at a microscopic level. If the dynamics of each particle was know it should be possible in principle to determine the macroscopic dynamics of the fluid from the collective motion of the particles.

II. STATE SPACE DYNAMICS

Assuming the dynamics a particle is defined by the acceleration and that the acceleration of the particles is given by the potential ϕ such that $\vec{a} = \vec{\nabla} \cdot \phi$, then the equations of motion become

$$\frac{d}{dt}\begin{pmatrix}\vec{x} \\ \dot{\vec{x}}\end{pmatrix} = \begin{pmatrix}\dot{\vec{x}} \\ -\vec{\nabla}\phi\end{pmatrix}$$

For a system of n particles where the i^{th} particle is located at $\vec{x}_i$, the potential can in principle depend on the location of all of the particles, such that the potential for the i^{th} particle is $\phi_i = \phi(\vec{x}_i, \vec{x}_1, \vec{x}_2, ...\vec{x}_i..., \vec{x}_{n-1}, \vec{x}_n)$. Then for this system the equations of motion for each particle is given by the system of equations

$$\frac{d}{dt}\begin{pmatrix}\vec{x}_i \\ \dot{\vec{x}}_i\end{pmatrix} = \begin{pmatrix}\dot{\vec{x}}_i \\ -\partial_{\vec{x}_i}\phi_i\end{pmatrix} \tag{1}$$

Moving over to a state space description of the system, the state space has six coordinates given by the orthogonal coordinate vectors $\vec{x}$ and $\dot{\vec{x}}$. Given a time dependent density distribution defined over state space $\sigma = \sigma(\vec{x}, \dot{\vec{x}}, t)$, such that the total mass at the time t is $M(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(\vec{x}, \dot{\vec{x}}, t) d^3x d^3\dot{x}$. The system of discrete particles described by equation (1) then be described by the distribution

$$\sigma(\vec{x}, \dot{\vec{x}}, t) = m \sum_{i=0}^{n} \delta(\vec{x} - \vec{x}_i) \cdot \delta(\dot{\vec{x}} - \dot{\vec{x}}_i)$$

where m is the mass of the particles, and $\delta(\vec{x})$ is the Dirac delta distribution in 3-space. For this system the six component state space velocity is $\vec{v} = \left\langle \dot{\vec{x}}, -\vec{\nabla}\phi(\vec{x}, \rho, t) \right\rangle$, where the potential ϕ is defined as $\phi(\vec{x}_i, \rho, t) = \phi_i$ and $\rho(\vec{x}, t) = \int_{-\infty}^{\infty} \sigma(\vec{x}, \dot{\vec{x}}, t) d^3\dot{x}$. Assuming the number of particles in the distribution is constant, then $\frac{dM}{dt} = 0$. Since there are no sources or sinks for the particles, the density distribution is constrained by the continuity equation

$$\frac{d}{dt}\int_V \sigma(\vec{x}, \dot{\vec{x}}, t) d^3x d^3\dot{x} + \oint_{\partial V} \sigma(\vec{x}, \dot{\vec{x}}, t)\vec{v} \cdot d\vec{a} = 0$$

where V is an arbitrary volume in state space, ∂V is the surface of the arbitrary volume, $\vec{v}$ is the state space velocity and $d\vec{a}$ is a surface element in state space. Rearranging the terms and using the divergence theorem the continuity equation can be rewritten in the form

$$\int_V \left(\partial_t \sigma + \vec{\nabla} \cdot (\sigma \vec{v})\right) d^3x d^3\dot{x} = 0$$

Since the integral equals zero over any arbitrary volume V, then the integrand must be zero

$$\partial_t \sigma + \vec{\nabla} \cdot (\sigma \vec{v}) = 0$$

Finally the independence of $\vec{x}$ and $\dot{\vec{x}}$ can be used to rewrite the continuity equation in the final form, in this case it is written using Einstein notation

$$\partial_t \sigma + \dot{x}_i \partial_{x_i} \sigma - (\partial_{x_i}\phi)\, \partial_{\dot{x}_i} \sigma = 0 \tag{2}$$

While the derivation of equation (2) was motivated using a discrete collection of particles the equation is not restricted to systems of discrete particles. As long as the dynamics in state space is determined by $\vec{v} = \left\langle \dot{\vec{x}}, \vec{\nabla}\phi(\vec{x}, \rho, t) \right\rangle$ and there are no sources or sinks for the state space density, then any state space density function or distribution is described by equation (2).

Assuming the motion of individual particles, in a fluid described by fluid mechanics, is described by equation (1) and assuming that the number and mass of the particles is invariant, then the dynamics of the state space distribution is described by equation (2). If these assumptions are true for any fluid, then equation (2) must be capable of reproducing the behavior of fluid mechanics.

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III. FLUID MECHANICS MASS EQUATION: CONSERVATION OF MASS

The state space density function σ is defined such that $M = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(\vec{x},\dot{\vec{x}},t)d^3xd^3\dot{x}$. Define the mass density ρ as $\rho(\vec{x},t) = \int_{-\infty}^{\infty}\sigma(\vec{x},\dot{\vec{x}},t)d^3\dot{x}$. Also define the bulk velocity $\vec{u}$ as $u_i = \frac{1}{\rho(\vec{x},t)}\int_{-\infty}^{\infty}x_i\sigma(\vec{x},\dot{\vec{x}},t)d^3\dot{x} = \int_{-\infty}^{\infty}x_i\frac{\sigma}{\rho}d^3\dot{x}$. The integral of equation (2) over velocity space is

$$\int_{-\infty}^{\infty}\left(\partial_t\sigma + \dot{x}_i\partial_{x_i}\sigma - \left(\partial_{x_i}\phi\right)\partial_{\dot{x}_i}\sigma\right)d^3\dot{x} = 0$$

Splitting up the integral, pulling out factors and operations that are independent of $\dot{x}_i$ puts the equation into a form where some terms can be evaluated. Then applying the divergence theorem and evaluating the simplified integrals results in the equation

$$\partial_t\rho + \partial_{x_i}\left(u_i\rho\right) - \left(\partial_{x_i}\phi\right)\oint\sigma da_i = 0$$

where da_i is the i^{th} component of the surface element, and $\oint\sigma da_i$ is the surface integral over all of velocity space. Assuming $\sigma(\vec{x},\dot{\vec{x}},t)$ drops to zero faster than $|\dot{\vec{x}}|^2$ then the surface integral converges to zero. Given the surface integral does converge to zero, the resulting equation is $\partial_t\rho + \partial_{x_i}\left(u_i\rho\right) = 0$. When written in vector notation it becomes

$$\partial_t\rho + \vec{\nabla}\cdot(\vec{u}\rho) = 0 \tag{3}$$

which is the conservation of mass equation from fluid mechanics.

IV. CONSERVATION OF MOMENTUM

To get an equation for the conservation of momentum, multiply equation (2) by $\dot{\vec{x}}$ before integrating over velocity space.

$$\int_{-\infty}^{\infty}\dot{x}_i\left(\partial_t\sigma + \dot{x}_j\partial_{x_j}\sigma - \left(\partial_{x_j}\phi\right)\partial_{\dot{x}_j}\sigma\right)d^3\dot{x} = 0$$

After simplifying the equation, applying the product rule and divergence theorem it can be written in the form

$$\partial_t\left(u_i\rho\right) + \partial_{x_j}\left(\int_{-\infty}^{\infty}\dot{x}_i\dot{x}_j\sigma d^3\dot{x}\right) - \left(\partial_{x_j}\phi\right)\oint\dot{x}_i\sigma da_j + \left(\partial_{x_j}\phi\right)\rho\delta_{ij} = 0$$

where $\oint\dot{x}_i\sigma da_j$ is a surface integral over all of velocity space and δ_{ij} is the Kronecker delta function. Assuming $\sigma(\vec{x},\dot{\vec{x}},t)$ drops to zero faster than $|\dot{\vec{x}}|^3$ then the surface integral converges to zero. Given the surface integral does converge to zero

$$\partial_t\left(u_i\rho\right) + \partial_{x_j}\left(\int_{-\infty}^{\infty}\dot{x}_i\dot{x}_j\sigma d^3\dot{x}\right) + \rho\partial_{x_i}\phi = 0 \tag{4}$$

A. Integrating the velocity tensor product

The velocity tensor product term is $\int_{-\infty}^{\infty}\dot{x}_i\dot{x}_j\sigma d^3\dot{x}$. Since the bulk velocity is defined as $u_i = \int_{-\infty}^{\infty}\dot{x}_i\frac{\sigma}{\rho}d^3\dot{x}$ let

$$\int_{-\infty}^{\infty}\dot{x}_i\dot{x}_j\sigma d^3\dot{x} = \int_{-\infty}^{\infty}\dot{x}_i\frac{\sigma}{\rho}d^3\dot{x}\int_{-\infty}^{\infty}\dot{x}_j\sigma d^3\dot{x} - \int_{-\infty}^{\infty}\dot{x}_i\frac{\sigma}{\rho}d^3\dot{x}\int_{-\infty}^{\infty}\dot{x}_j\sigma d^3\dot{x} + \int_{-\infty}^{\infty}\dot{x}_i\dot{x}_j\sigma d^3\dot{x}$$

Evaluating the first term and combining the other two terms produces the equation

$$\int_{-\infty}^{\infty}\dot{x}_i\dot{x}_j\sigma d^3\dot{x} = \rho u_i u_j + \frac{1}{2}\left(\int_{-\infty}^{\infty}\left(\dot{x}_i - u_i\right)\dot{x}_j\sigma d^3\dot{x} + \int_{-\infty}^{\infty}\left(\dot{x}_j - u_j\right)\dot{x}_i\sigma d^3\dot{x}\right)$$

If we define a symmetric tensor A such that $A_{ij} = -\frac{1}{2}\left(\int_{-\infty}^{\infty}\left(\dot{x}_i - u_i\right)\dot{x}_j\sigma d^3\dot{x} + \int_{-\infty}^{\infty}\left(\dot{x}_j - u_j\right)\dot{x}_i\sigma d^3\dot{x}\right)$, then the integral of the velocity tensor product simplifies to the equation

$$\int_{-\infty}^{\infty}\dot{x}_i\dot{x}_j\sigma d^3\dot{x} = \rho u_i u_j - A_{ij}$$

Let $p = -\frac{1}{3}A_{ii}$ and define the traceless tensor B such that $B_{ij} = A_{ij} - \frac{1}{3}A_{kk}\delta_{ij}$, so the velocity tensor product term can be written as

$$\int_{-\infty}^{\infty}\dot{x}_i\dot{x}_j\sigma d^3\dot{x} = \rho u_i u_j + p\delta_{ij} - B_{ij}$$

B. Fluid mechanics momentum equation: Navier-Stokes equations

Substituting the integral of the velocity tensor product into equation (4)

$$\partial_t\left(u_i\rho\right) + \partial_{x_j}\left(\rho u_i u_j + p\delta_{ij} - B_{ij}\right) + \rho\partial_{x_i}\phi = 0$$

Expand the partial derivatives involving the bulk velocity and canceling terms leading with u_i by substituting in equation (3). Then the equation can then be written in vector notation as

$$\rho\left(\partial_t\vec{u} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = -\vec{\nabla}p + \vec{\nabla}B - \rho\vec{\nabla}\phi \tag{5}$$

Assuming that for a reasonable model of a fluid the tensor A_{ij} evaluates to be the total stress tensor σ_{ij}, then equation (5) is the Navier-Stokes equations.

V. CONSERVATION OF ENERGY

The total energy of the system is $E_{\text{tot}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{2}\dot{x}_i^2 + \phi(\vec{x},\rho)\right)\sigma(\vec{x},\dot{\vec{x}},t)d^3xd^3\dot{x}$. Let us define $\vec{v} = \dot{\vec{x}} - \vec{u}$, so then $\dot{x}_i^2 = v_i^2 + 2\dot{x}_iu_i - u_i^2$. In order to derive an equation for the conservation of energy equation in position space, equation (2) must be multiplied by $\frac{1}{2}\dot{x}_i^2 + \phi$ before integrating. To make the derivation of the conservation of energy simpler we can use the linearity of integration to break the conservation of energy equation into a kinetic energy component and a potential energy component.

A. The kinetic energy component

Multiplying equation (2) by $\frac{1}{2}\dot{x}_i^2$ then integrating over velocity space, the resulting relation is

$$\int_{-\infty}^{\infty}\frac{1}{2}\dot{x}_i^2\left(\partial_t\sigma + \dot{x}_j\partial_{x_j}\sigma - \left(\partial_{x_j}\phi\right)\partial_{\dot{x}_j}\sigma\right)d^3\dot{x} = 0$$

After simplifying the equation, applying the product rule and divergence theorem, it can be written in the form

$$\begin{aligned}&\partial_t\left(\int_{-\infty}^{\infty}\frac{1}{2}\dot{x}_i^2\sigma d^3\dot{x}\right) + \partial_{x_j}\left(\int_{-\infty}^{\infty}\frac{1}{2}\dot{x}_i^2\dot{x}_j\sigma d^3\dot{x}\right)\\ &- \left(\partial_{x_j}\phi\right)\left(\oint\left(\frac{1}{2}\dot{x}_i^2\sigma\right)da_j - \int_{-\infty}^{\infty}\sigma\dot{x}_jd^3\dot{x}\right) = 0\end{aligned}$$

where $\oint\left(\frac{1}{2}\dot{x}_i^2\sigma\right)da_j$ is a surface integral over all of velocity space. Assuming $\sigma(\vec{x},\dot{\vec{x}},t)$ drops to zero faster than $|\dot{\vec{x}}|^4$ then the surface integral converges to zero. Evaluating integrals and rearranging the equation given the surface integral does converge to zero, results in the equation

$$\partial_t\left(\int_{-\infty}^{\infty}\frac{1}{2}\dot{x}_i^2\sigma d^3\dot{x}\right) + \partial_{x_j}\left(\int_{-\infty}^{\infty}\frac{1}{2}\dot{x}_i^2\dot{x}_j\sigma d^3\dot{x}\right) + u_j\rho\partial_{x_j}\phi = 0$$

After expanding the equation by using the definition of v_i and rearranging, use the result $\int_{-\infty}^{\infty}\dot{x}_i\dot{x}_j\sigma d^3\dot{x} = \rho u_iu_j + p\delta_{ij} - B_{ij}$ from the momentum equation derivation to write the equation in the form

$$\begin{aligned}&\partial_t\left(\int_{-\infty}^{\infty}\frac{1}{2}v_i^2\sigma d^3\dot{x} + \frac{1}{2}u_i^2\rho\right) + u_j\rho\partial_{x_j}\phi\\ &+ \partial_{x_j}\left(\int_{-\infty}^{\infty}\frac{1}{2}v_i^2\dot{x}_j\sigma d^3\dot{x} + \frac{1}{2}u_i^2u_j\rho + pu_j - u_iB_{ij}\right) = 0\end{aligned} \tag{6}$$

B. The potential energy component

Multiplying equation (2) by ϕ and then integrating over velocity space, the resulting relation is

$$\int_{-\infty}^{\infty}\phi\left(\partial_t\sigma + \dot{x}_j\partial_{x_j}\sigma - \left(\partial_{x_j}\phi\right)\partial_{\dot{x}_j}\sigma\right)d^3\dot{x} = 0$$

After rearranging and simplifying use the result that $\int_{-\infty}^{\infty}\partial_{\dot{x}_j}\sigma d^3\dot{x} = \oint\sigma da_j = 0$, from the derivation for the conservation of mass equation, to get

$$\phi\partial_t\rho + \phi\partial_{x_j}\left(u_j\rho\right) = 0$$

Applying the product rule produces the form

$$\partial_t\left(\phi\rho\right) - \rho\partial_t\phi + \partial_{x_j}\left(u_j\rho\phi\right) - u_j\rho\partial_{x_j}\phi = 0 \tag{7}$$

C. Total energy density dynamics

The potential ϕ can be split into two components, the internal potential ϕ_{in} and a potential due to the external environment ϕ_{ext}, such that $\phi = \phi_{\text{in}} + \phi_{\text{ext}}$. Let us define two energy densities, the internal energy density e and the external energy density η, such that $e(\vec{x},t) = \int_{-\infty}^{\infty}\left(\frac{1}{2}v_i^2 + \phi_{\text{in}}\right)\sigma d^3\dot{x}$ and $\eta(\vec{x},t) = \int_{-\infty}^{\infty}\left(\frac{1}{2}u_i^2 + \phi_{\text{ext}}\right)\sigma d^3\dot{x} = \left(\frac{1}{2}u_i^2 + \phi_{\text{ext}}\right)\rho$. From these definitions the total energy is then $E_{\text{tot}} = \int_{-\infty}^{\infty}\left(e + \eta\right)d^3x$. Adding the two energy components from equation (6) and equation (7) produces the equation

$$\begin{aligned}&\partial_t\left(\int_{-\infty}^{\infty}\frac{1}{2}v_i^2\sigma d^3\dot{x} + \frac{1}{2}u_i^2\rho\right) + u_j\rho\partial_{x_j}\phi\\ &+ \partial_{x_j}\left(\int_{-\infty}^{\infty}\frac{1}{2}v_i^2\dot{x}_j\sigma d^3\dot{x} + \frac{1}{2}u_i^2u_j\rho + pu_j - u_iB_{ij}\right)\\ &+ \partial_t\left(\phi\rho\right) - \rho\partial_t\phi + \partial_{x_j}\left(u_j\rho\phi\right) - u_j\rho\partial_{x_j}\phi = 0\end{aligned}$$

After simplifying and splitting ϕ into its two components and evaluating terms using the energy density definitions the equation can be written as

$$\begin{aligned}&\partial_t\left(e + \eta\right) - \rho\partial_t\phi + \partial_{x_j}\left(\int_{-\infty}^{\infty}\frac{1}{2}v_i^2\dot{x}_j\sigma d^3\dot{x}\right.\\ &\left.+ u_j\phi_{\text{in}}\rho + \eta u_j + pu_j - u_iB_{ij}\right) = 0\end{aligned}$$

Using the definition of v_i and applying the definition internal energy density the relation is

$$\begin{aligned}&\partial_t\left(e + \eta\right) + \partial_{x_j}\left(eu_j + \eta u_j\right)\\ &+ \partial_{x_j}\left(\int_{-\infty}^{\infty}\frac{1}{2}v_i^2v_j\sigma d^3\dot{x} + pu_j - u_iB_{ij}\right) - \rho\partial_t\phi = 0\end{aligned}$$

Defining the vector $\vec{C}$ as $C_j = \int_{-\infty}^{\infty}\frac{1}{2}v_i^2v_j\sigma d^3\dot{x}$ and using it to simplify the equation results in the form

$$\begin{aligned}&\partial_t\left(e + \eta\right) + \partial_{x_j}\left(eu_j + \eta u_j\right)\\ &+ \partial_{x_j}\left(C_j + pu_j - u_iB_{ij}\right) - \rho\partial_t\phi = 0\end{aligned} \tag{8}$$

Equation (8) describes the dynamics of the total energy density. Using equation (5) an equation for the external energy density can be derived. Which when combined with equation (8) will produce an equation for the internal energy density.

D. External energy density dynamics

To generate an equation for the external energy density, let us multiply equation (5) by u_i.

$$u_i \left(\rho \partial_t u_i + \rho u_j \partial_{x_j} u_i + \partial_{x_i} p - \partial_{x_j} B_{ij} + \rho \partial_{x_i} \phi\right) = 0$$

Rearranging and applying the product rule, the equation becomes

$$\partial_t \left(\frac{1}{2} u_i^2 \rho\right) - \frac{1}{2} u_i^2 \partial_t \rho + \partial_{x_j} \left(\frac{1}{2} u_i^2 \rho u_j\right) - \frac{1}{2} u_i^2 \partial_{x_j} (\rho u_j) + u_i \left(\partial_{x_i} p - \partial_{x_j} B_{ij} + \rho \partial_{x_i} \phi\right) = 0$$

After collecting terms of $\frac{1}{2}u_i^2$, adding a potential energy component $\partial_t (\rho\phi_{\text{ext}} - \rho\phi_{\text{ext}}) + \partial_{x_j} (u_j\rho\phi_{\text{ext}} - u_j\rho\phi_{\text{ext}}) = 0$, applying the product rule and using the definition of η the equation can put in the form

$$\partial_t \eta - \rho \partial_t \phi_{\text{ext}} - \left(\frac{1}{2} u_i^2 + \phi_{\text{ext}}\right) \left(\partial_t \rho + \partial_{x_j} (\rho u_j)\right) + \partial_{x_j} (\eta u_j) - u_j \rho \partial_{x_j} \phi_{\text{ext}} + u_i \left(\partial_{x_i} p - \partial_{x_j} B_{ij} + \rho \partial_{x_i} \phi\right) = 0$$

Canceling terms by substituting in equation (3) and relabeling contracted indices results in the equation

$$\begin{aligned} &\partial_t \eta + \partial_{x_j} (\eta u_j) - \rho \partial_t \phi_{\text{ext}} \\ &+ u_j \partial_{x_j} p - u_i \partial_{x_j} B_{ij} + u_j \rho \partial_{x_j} \phi_{\text{in}} = 0 \end{aligned} \tag{9}$$

E. Fluid mechanics energy equation: Internal energy equation

To remove the external energy density from equation (8) subtract equation (9) from it, resulting in the equation

$$\begin{aligned} &\partial_t (e + \eta) + \partial_{x_j} (e u_j + \eta u_j) - \rho \partial_t \phi \\ &+ \partial_{x_j} (C_j + p u_j - u_i B_{ij}) \\ &- \partial_t \eta - \partial_{x_j} (\eta u_j) + \rho \partial_t \phi_{\text{ext}} \\ &- u_j \partial_{x_j} p + u_i \partial_{x_j} B_{ij} - u_j \rho \partial_{x_j} \phi_{\text{in}} = 0 \end{aligned}$$

After applying the product rule and simplifying, the equation can be written as

$$\begin{aligned} &\partial_t e + \partial_{x_j} (e u_j) - \rho \partial_t \phi_{\text{in}} - u_j \rho \partial_{x_j} \phi_{\text{in}} \\ &+ \partial_{x_j} C_j + p \partial_{x_j} u_j - B_{ij} \partial_{x_j} u_i = 0 \end{aligned}$$

rearranging the equation and putting it in vector notation, the equation then can be written in the from

$$\begin{aligned} &\partial_t e + \vec{\nabla} \cdot (e \vec{u}) = \rho \partial_t \phi_{\text{in}} + \rho \vec{u} \cdot \vec{\nabla} \phi_{\text{in}} \\ &- \vec{\nabla} \cdot \vec{C} - p \vec{\nabla} \cdot \vec{u} + B \cdot \vec{\nabla} \vec{u} \end{aligned} \tag{10}$$

Assuming that for a reasonable model of a fluid the tensor A_{ij} evaluates to be the total stress tensor σ_{ij} and the vector C_j reduce a term including the gradient of the temperature, then equation (10) is a form of the fluid mechanics internal energy equation.

VI. FLUID MODEL: MAXWELL-BOLTZMANN DISTRIBUTION

Assuming the speed of particles in a fluid follow a Maxwell-Boltzmann distribution, then the velocity distribution of the fluid is a normalized Gaussian distribution in three dimensional velocity space. For a fluid element with a non-zero bulk velocity assume that the velocity distribution is simply a normalized Gaussian with a non-zero mean value. Given these assumptions the velocity distribution at any point is $\sigma(\vec{x}, \dot{\vec{x}})/\rho(\vec{x}) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m(\dot{x}_i - u_i)^2}{2kT}}$, where m is the mass of the particles the fluid is made from, k is Boltzmann's constant and $T = T(\vec{x}, t)$ is the thermodynamic temperature. Also assume that the potential ϕ only explicitly depends on $\vec{x}$ and ρ. So in this model $\phi = \phi(\vec{x}, \rho)$ and the state space density distribution is,

$$\sigma(\vec{x}, \dot{\vec{x}}) = \rho(\vec{x}) \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m(\dot{x}_i - u_i)^2}{2kT}} \tag{11}$$

As defined earlier the tensor A is

$$A_{ij} = -\frac{1}{2} \left(\int_{-\infty}^{\infty} (\dot{x}_i - u_i)\, \dot{x}_j \sigma d^3\dot{x} + \int_{-\infty}^{\infty} (\dot{x}_j - u_j)\, \dot{x}_i \sigma d^3\dot{x} \right)$$

When i is not equal to j the integral is $A_{ij} = 0$, when i is equal to j the integral is $A_{ij} = -kT\rho/m$, so the full tensor is $A_{ij} = -kT\frac{\rho}{m}\delta_{ij}$. Using the ideal gas law $PV = NkT$, since $\rho/m = N/V$ where N is the number of particles and V is the volume, then $A_{ij} = -P\delta_{ij}$, $p = -\frac{1}{3}A_{ii} = P$ and $B_{ij} = A_{ij} - \frac{1}{3}A_{kk}\delta_{ij} = 0$.

The vector $\vec{C}$ is defined as $C_j = \int_{-\infty}^{\infty} \frac{1}{2} v_i^2 v_j \sigma d^3\dot{x}$, evaluating the integral results in the equation $C_j = 0$. So given the state space density function (11) the equations of fluid mechanics derived from equation (2) are

$$\partial_t \rho + \vec{\nabla} \cdot (\vec{u}\rho) = 0$$

$$\rho \left(\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u}\right) = -\vec{\nabla} P - \rho \vec{\nabla} \phi$$

$$\partial_t e + \vec{\nabla} \cdot (e\vec{u}) = \rho \vec{u} \cdot \vec{\nabla} \phi_{\text{in}} - P \vec{\nabla} \cdot \vec{u}$$

These equations are equal to the conservation of mass, internal energy density and Navier-Stokes equations in the case of inviscid flow.